mum launch azimuths, since for this launch time the optimum azimuth is due-east. The entire trajectory is seen to lie in the reference plane, and where the trajectory terminates, the reference plane is extended by the dashed curve. For the other two launch times, the reference plane is translated and drawn through the launch site in order to yield a visualization of two-dimensional motion.

Some understanding of the variation of launch azimuth can be obtained from this figure. Consider, for example, the trajectories shown for $\Delta t_L = +1$ hr. For the due-east launch azimuth, the trajectory is "S" shaped. There are two "turnings" or bendings of the trajectory. Near the launch site, the flight plane is turned southerly to bring the motion toward the desired plane. Towards the injection point, a second turning is necessary to bring the velocity vector as well as the position vector into the plane.

Optimization of the launch azimuth is seen to alleviate the necessity for the bending around the launch site. Thus, the bending of the trajectory, i.e., its three-dimensionality, can be equated roughly with energy expenditure.

This would leave one with a final conjecture. The initial bending of the plane seen for the due-east launch azimuth is necessary to direct the motion toward the desired plane, and if this were not done, burnout would occur at some point relatively remote from the reference plane. But if a coast were introduced at a sufficiently high energy level to avoid excessive descent, and if the coast were of a sufficiently long duration to allow a displacement of the order of 180°, the vehicle would approach automatically the reference plane from the other side of the earth. This may improve performance in much the same manner as did optimization of the launch azimuth.

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Investigation of Maximum Stresses in Long, Pressurized, Cylindrical Shells

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Equations are derived for the slope, deflection, and maximum stresses in a cylindrical shell subjected to edge shears and moments, including the simultaneous action of axial loads. A sample problem is given in which the maximum stresses obtained by the refined equations are compared to those obtained by approximate methods currently used in industry in which the axial load restraints on shell rotations and deflections are neglected. A 100-in. radius cylinder with a wall thickness of 0.5 in. and an internal pressure of 1000 psi was selected arbitrarily as a basis for comparison. The maximum hoop stress was found to be 213,770 psi by the standard method and 200,000 psi by the method developed in this paper. The reduction in maximum hoop stress is due to the restraining influence of the axial loads on hoop displacements of the shell. Where warranted, the use of the refined method generally will give lower calculated maximum stresses and result in a lighter-weight shell design.

Introduction

A refined analysis is presented which accounts for the axial load effect on the maximum stresses in a long, pressurized, cylindrical shell. A comparison between the refined analysis and the standard analysis currently used in industry is given. This paper reports the first phase of an investigation that ultimately will establish parameters defining the relative merits of each analysis.

Discussion

Figure 1 shows a cylindrical shell constituting the central portion of a pressure vessel subjected to internal pressure p and three edge loads at the left end: moment M, shear V, and axial load N. Values of M and V are obtained by performing a compatibility analysis between the cylindrical shell and the adjoining structure. Wei[1] has presented the theory and formulas used in establishing the compatibility equations and shows their application in stress-analyzing solid propellant rocket cases. In addition, formulas for obtaining the maximum meridional (axial) and hoop (circumferential) stresses in the cylindrical shell (Fig. 1) were derived by Wei. These formulas are repeated in Tables 1 and 2 for convenience.

Hetenyi[2] points out that current stress analysis practice is to use formulas that neglect the effect of the axial load N on shell rotations and deflections due to edge loads M and V. Maximum stresses calculated using these standard formulas in the compatibility equations compare reasonably well with experimental stresses for rocket case (or pressure vessel) sizes currently used in industry.[1] It is conceivable, however, that, in the near future, vessel sizes will be of such diameter and wall thickness that a more refined analysis, i.e., considering the effect of the axial load on shell slope, deflection, and stress, must be used.

A set of formulas has been derived which accounts for the axial load effect and which gives 1) the slope and deflection at the edge of a cylindrical shell due to edge moment and shear loads, 2) the location of points of zero slope on the meridional and hoop stress curves, and 3) the magnitude of the meridional and hoop stresses. The formulas are based on the similarity between a cylindrical shell and a beam on

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Table 1 Equations for edge rotation and deflection

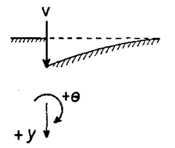
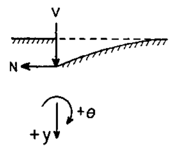
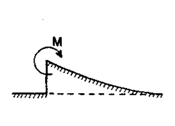
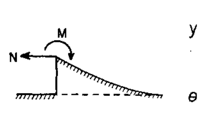
N-Neglected		N-Considered	
	$y = \frac{2V}{\lambda Z}$ $\theta = -\frac{2V}{Z}$		$y = \frac{2V\sqrt{1+\Delta}}{\lambda(Z+2N)}$ $\theta = -\frac{2V}{Z+2N}$
	$y = -\frac{2M}{Z}$ $\theta = \frac{4M\lambda}{Z}$		$y = -\frac{2M}{Z+2N}$ $\theta = \frac{4M\lambda\sqrt{1+\Delta}}{Z+2N}$

Table 2 Equations for meridional and hoop stresses

N-Neglected	N-Considered	
	N<Z	N>Z
Meridional Stress Curve Zero Slope Locations: $x = \frac{1}{\lambda}\tan^{-1}\frac{W}{W-1}$ Meridional Stress $= \pm\frac{6}{t^2}\left[M(A) - \frac{V}{\lambda}(B)\right] + \left[\frac{A_N}{A_G}\right]\frac{pR}{2t}$	Meridional Stress Curve Zero Slope Locations: $x = \frac{1}{\beta}\tan^{-1}\beta G$ Meridional Stress $= \pm\frac{6e^{-\alpha x}}{t^2}\left[M\cos\beta x - \frac{H}{\beta}\sin\beta x\right] + \frac{N}{t}$	Meridional Stress Curve Zero Slope Locations: $x = \frac{1}{\bar{\beta}}\mathrm{Tanh}^{-1}\bar{\beta}G$ Meridional Stress $= \pm\frac{6e^{-\alpha x}}{t^2}\left[M\cosh\bar{\beta}x - \frac{H}{\bar{\beta}}\sinh\bar{\beta}x\right] + \frac{N}{t}$
Hoop Stress Curve Zero Slope Locations: $x = \frac{1}{\lambda}\tan^{-1}\frac{W(1-S)-1}{S-W(1+S)}$ Hoop Stress $= \frac{pR}{t} - \frac{2V\lambda R(D)}{t} + \frac{2M\lambda^2 R(C)}{t}$ $\pm\frac{6\mu}{t^2}\left[M(A) - \frac{V}{\lambda}(B)\right]$	Hoop Stress Curve Zero Slope Locations: $x = \frac{1}{\beta}\tan^{-1}\frac{F-\alpha J}{\frac{\alpha F}{\beta}+\beta J}$ Hoop Stress $= \frac{R}{t}\left[p + \frac{e^{-\alpha x}}{\lambda^2(\frac{1}{2}+\Delta)}(J\cos\beta x + \frac{F}{\beta}\sin\beta x)\right]$	Hoop Stress Curve Zero Slope Locations: $x = \frac{1}{\bar{\beta}}\mathrm{Tanh}^{-1}\frac{F-\alpha J}{\frac{\alpha F}{\bar{\beta}}-\bar{\beta}J}$ Hoop Stress $= \frac{R}{t}\left[p + \frac{e^{-\alpha x}}{\lambda^2(\frac{1}{2}+\Delta)}(J\cosh\bar{\beta}x + \frac{F}{\bar{\beta}}\sinh\bar{\beta}x)\right]$
$A = e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$ $B = e^{-\lambda x}(\sin\lambda x)$ $C = e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$ $D = e^{-\lambda x}(\cos\lambda x)$ $W = \frac{V}{2M\lambda}$, $S = \pm\frac{3\mu}{\sqrt{3(1-\mu^2)}}$	$\alpha = \lambda\sqrt{1+\Delta}$ $\beta = \lambda\sqrt{1-\Delta}$ $\bar{\beta} = \lambda\sqrt{\Delta-1}$ $F = -V\lambda^2(\Delta\lambda^2 \pm \frac{3\mu}{Rt}) + M\alpha\lambda^2\left[-\lambda^2 \pm \frac{3\mu}{Rt}(1-2\Delta)\right]$ $G = \frac{V+4M\alpha\Delta}{V\alpha - 2M\lambda^2(1-2\Delta^2)}$ $H = \frac{V+M\alpha(2\Delta-1)}{1+2\Delta}$ $J = -V\lambda^2\alpha + M\left[\lambda^4 \pm \frac{3\mu}{Rt}\lambda^2(1+2\Delta)\right]$	

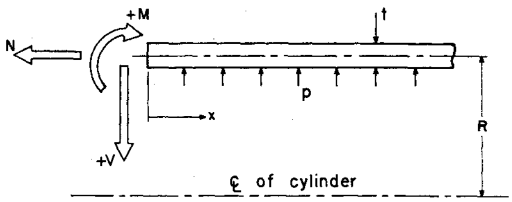


Fig. 1 Long cylindrical shell subjected to internal pressure and edge loads

an elastic foundation and are an extension of the equations given in Ref. 2.

Table 1 lists the standard equations for edge rotation and deflection (neglecting N) and the equivalent formulas for the case when N is considered. In these formulas,

$$N = (A_n/A_g) \times (pR/2)$$

where

A_n = cross-sectional area of cylinder effective in producing axial loads in cylinder wall
A_g = full cross-sectional area of cylinder
A_n/A_g = axial load area factor

In physical terms, A_n/A_g represents the percentage of the full axial load $pR/2$ which is carried by the cylinder wall. In the case of a closed-end cylinder, $A_n/A_g = 1.0$; for a cylinder with ends sealed off by a floating piston, $A_n/A_g = 0$. In the hydrotest of a solid propellant rocket case where the nozzle ports are closed off by floating pistons, the axial load area factor will lie between the extreme limits of 1.0 and zero. The two sets of formulas in Table 1 are similar to each other in form except that the equations for the N-considered case contain two additional terms, N and Δ, where

$$\Delta = \frac{N}{Z} \qquad Z = \frac{K}{\lambda^2} \qquad K = \frac{Et}{R^2}$$

$$\lambda = \frac{[3(1-\mu^2)]^{1/4}}{(Rt)^{1/2}}$$

Table 3 Typical computer output sheet for cylinder with axial load area factor of 1.0 and N-neglected

P	R	T	L	M	V	N	μ	λ	Δ	
1000.	100.00	0.500	1.00	12859.	4675.	50000.	0.30	0.182	0.	N-neglected

Deflection	Rotation	E	K	Z
−1.212E-04	5.323E-06	30.0E 06	1.500E 03	4.539E 04

Maximum meridional stress	Location of maximum merid. stress	Maximum hoop stress	Location of maximum hoop stress
408616.	0.	213772.	14.538

x	Meridional stress, outer fiber	Meridional stress, inner fiber	Hoop stress, outer fiber	Hoop stress, inner fiber
0.	−208616.	408616.	−62548.	122621.
1.3753	−73418.	273418.	−13028.	91023.
2.7505	25466.	174534.	37746.	82466.
4.1258	92699.	107301.	84338.	88718.
5.5010	134188.	65812.	123857.	103344.
6.8763	156024.	43976.	155239.	121624.
8.2516	163814.	36186.	178631.	140342.
9.6268	162327.	37673.	194899.	157503.
11.0021	155357.	44643.	205263.	172048.
12.3773	145741.	54259.	211036.	183592.
13.7526	135455.	64545.	213464.	192191.
15.1278	125765.	74235.	213625.	198166.
16.5031	117380.	82620.	212397.	201969.
17.8784	110601.	89399.	210445.	204084.
19.2536	105458.	94542.	208244.	204969.
20.6289	101807.	98193.	206107.	205022.
22.0041	99417.	100583.	204216.	204566.
23.3794	98024.	101976.	202659.	203844.
24.7547	97371.	102629.	201455.	203032.
26.1299	97232.	102768.	200584.	202245.
27.5052	97416.	102584.	199998.	201549.
28.8804	97780.	102220.	199643.	200975.

Table 4 Typical computer output sheet for cylinder with axial load area factor of 1.0 and N-considered

P	R	T	L	M	V	N	μ	λ	Δ	N-considered
1000.	100.00	0.500	1.00	12859.	6778.	50000.	0.30	0.182	1.102	

Deflection	Rotation	E	K	Z
−1.901E-05	−9.141E-06	30.0E 06	1.500E 03	4.539E 04

Maximum meridional stress	Location of maximum merid. stress	Maximum hoop stress	Location of maximum hoop stress
408616.	0.	200000.	–membrane region–

x	Meridional stress, outer fiber	Meridional stress, inner fiber	Hoop stress, outer fiber	Hoop stress, inner fiber
0.	−208616.	408616.	−62579.	122591.
1.3753	−37545.	237545.	−2876.	79651.
2.7505	57428.	142572.	43899.	69442.
4.1258	107180.	92820.	80294.	75986.
5.5010	130648.	69352.	108458.	90069.
6.8763	139319.	60681.	130157.	106565.
8.2516	140059.	59941.	146814.	122779.
9.6268	136892.	63108.	159564.	137428.
11.0021	132112.	67888.	169298.	150031.
12.3773	126970.	73030.	176715.	160533.
13.7526	122106.	77894.	182357.	169094.
15.1278	117805.	82195.	186643.	175960.
16.5031	114154.	85846.	189894.	181402.
17.8784	111140.	88860.	192359.	185675.
19.2536	108700.	91300.	194225.	189005.
20.6289	106752.	93248.	195637.	191586.
22.0041	105214.	94786.	196705.	193576.
23.3794	104011.	95989.	197512.	195106.
24.7547	103075.	96925.	198122.	196278.
26.1299	102351.	97649.	198583.	197173.
27.5052	101793.	98207.	198931.	197855.
28.8804	101365.	98635.	199194.	198374.

When the value of N equals zero, i.e., there is no axial load in the shell, the equations for the N-considered case reduce to those for the standard case, i.e., N-neglected. General expressions for meridional and hoop stress in the cylinder of Fig. 1 are given in Table 2 for both cases N-neglected and N-considered. Expressions for hoop and meridional stresses for N-neglected (left-hand column of Table 2) are taken from Ref. 1 with changed notation for some terms and are reprinted in Table 2 for convenience and comparison with the N-considered case.

The location and magnitude of the maximum hoop and meridional stresses are the most important quantities of interest to the stress analyst and designer of pressure vessels. In particular, solid propellant rocket cases must be designed for minimum weight, and, hence, an accurate analytical determination of the maximum stresses must be made. The maximum stresses always will occur at one of the following three locations: 1) when x is equal to zero, 2) where the slope of the stress curve is zero, or 3) in the membrane region where the maximum stress equals pR/t.

In order to find the maximum stress, the stress at each of the foregoing locations is computed and compared. Locations on the stress curve where the slope is zero may be a maximum. In order to determine these zero-slope locations, the general formulas for stress were differentiated and equated to zero, resulting in trigonometric and hyperbolic expressions for the location of zero slope in terms of the cylinder constants and the edge loads M and V. The trigonometric expressions (arc tan) will give more than one location x for a given set of inputs, since the tangent function is periodic, with period π. In general, it will be necessary to check only the first and second locations for stresses, since locations beyond the first two will give stresses approximately equal to the membrane values obtained in item 3 in the foregoing. The arg tanh expressions in Table 2 for the case of N-considered ($N > Z$) will give only one location x for a given set of inputs, since the tanh function is single-valued. If the stress curve does not exhibit a point of zero slope for positive values of x, the maximum stress will occur either at the edge or in the membrane region, since imaginary and negative values of x are discarded.

Stresses in Table 2 for N-considered are dependent on the constants F, G, H, and J, which in turn are functions of the physical and geometric properties of the cylinder and the edge loads M and V. Two of these constants, F and J, contain the term $\pm\, 3\mu/Rt$. When the $+3\mu/Rt$ term is used in the formula for F, then the same sign ($+3\mu/Rt$) must be used in the formula for J. Similarly, when $-3\mu/Rt$ is used for F, the negative sign for this term must be used in calculating J. Thus two matched sets of F and J will be obtained. Each matched set should be tested separately for location of zero slope on the hoop stress curve and for the hoop stresses corresponding to these locations. Each matched set represents the hoop stress curve on one of the extreme fibers of the cylinder wall. Similarly, values of the hoop stress for the N-neglected case must be tested for locations given by the quantity $+S$ and $-S$ (see Table 2).

Procedure

The procedure for calculating maximum stresses in a cylinder is as follows:

1) Select the appropriate case, N-considered or N-neglected.

2) Perform a compatibility analysis between the cylinder and the mating element using the formulas for edge rotation and deflection from Table 1.

3) Determine the discontinuity shear V and moment M acting at the edge of the cylinder.

4) Calculate the location (or locations) of points of zero slope on the meridional and hoop stress curves, if any, using the equations in Table 2. Exercise care in calculating matched sets of the constants F and J.

5) Calculate the value of the stress at each of the foregoing locations using the equations in Table 2.

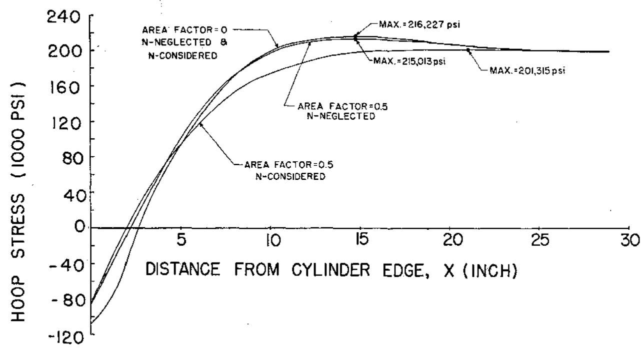


Fig. 2 Outer fiber hoop stress for area factors 0.0 and 0.5 with *N*-neglected and *N*-considered

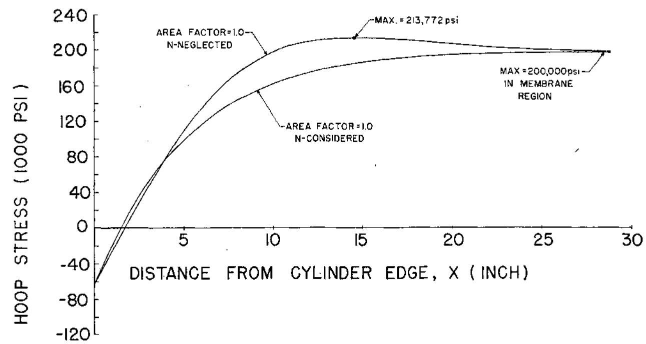


Fig. 3 Outer fiber hoop stress for area factor 1.0 with *N*-neglected and *N*-considered

6) Compare the stresses calculated in step 5 with the stresses at the edge ($x = 0$) and in the membrane region to determine the maximum value.

Application of Method

An investigation has been initiated with the ultimate objective of establishing parameters defining the relative accuracy of each analysis, N-considered and N-neglected. As a result, the standard analysis always could be used except in those cases where the parameters show a significant difference in maximum stresses. The formulas in Table 2 were programmed for digital computer solution. Various cylinders with various end conditions have been selected for study, and for each cylinder the maximum stresses are determined and compared for both N-considered and N-neglected.

An example of each analysis is given in Tables 3 and 4 and in Figs. 2 and 3 for a typical cylinder with one end fully fixed in a wall (built-in). The cylinder was selected arbitrarily to give a membrane hoop stress of 200,000 psi with pressure $p = 1000$ psi, radius $R = 100$ in., wall thickness $t = 0.50$ in., modulus of elasticity $E = 30{,}000{,}000$ psi, and Poisson's ratio $\mu = 0.3$.

Three values of the axial load area factor (A_n/A_g)—0.0, 0.5, and 1.0—were selected, and maximum stresses in each cylinder were computed by the program for N-considered and N-neglected, resulting in a total of six analyses.

Results

Table 3 shows a typical computer output sheet for the cylinder with axial load area factor of 1.0 and N-neglected. The edge loads were obtained initially from an analysis that set the final rotation and deflection of the cylindrical shell at the built-in edge equal to zero and used the formulas of Table 1 for N-neglected. The moment M was found to be 12,859 lb-in./in. of circumference, and the shear V was 4675 lb/in. Both M and V are used as program input together with p, R, t, μ, E, and the axial load area factor (designated as L on the output sheet).

Similarly, Table 4 shows an output sheet for the same cylinder as Table 3 but with N-considered. In this case, the edge loads were found using the rotation and deflection formulas of Table 1, N-considered, and M was found to be 12,859 lb-in./in. (the same as the foregoing), and V was 6778 lb/in. (higher than the N-neglected case).

Preliminary Findings

Figure 2 shows a plot of the outer fiber hoop stress vs the distance x from the edge of the cylinder for an axial load area factor of zero. When the area factor is zero, both N-considered and N-neglected analyses give the same stresses. In addition, Fig. 2 shows the outer fiber hoop stresses for an area factor of 0.5, and Fig. 3 shows the results for an area factor of 1.0. The most interesting finding was the effect on the outer fiber hoop stress for an area factor of 1.0. In the N-neglected analysis, the maximum hoop stress of 213,772 psi occurred in the outer fiber at $x = 14.538$ in., which is a point on the stress curve with zero slope as shown in Fig. 3. In the N-considered analysis, the maximum hoop stress of 200,000 psi occurred in the membrane region, there being no point of zero slope on the outer fiber hoop stress curve. This result indicates that the axial load restrains hoop displacements due to the edge loads and exerts a "straightening-out" effect on the cylinder wall. The reduction in hoop stress from 213,772 psi (N-neglected) to 200,000 psi (N-considered) shows that a reduction in the calculated value of the maximum hoop stress is possible (for the cylinder selected) using the refined analysis. The same trend is observed for the effect of the axial load on the meridional stresses (neglecting the magnitude of these stresses near the built-in edge of the arbitrarily selected cylinder). However, before any definite conclusions can be made, the more comprehensive study currently under way must be completed. Results and conclusions of this study will be published at a later date.

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[1] Wei, B. C. F., "Structural analysis of solid propellant rocket casings," ARS Preprint 1590-61 (February 1961).

[2] Hetenyi, M., *Beams on Elastic Foundation* (University of Michigan Press, Ann Arbor, Mich., 1946), Chap. VI, p. 138.